A MODULI SPACE OF MINIMAL AFFINE LAGRANGIAN SUBMANIFOLDS

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Abstract. It is proved that the moduli space of all connected compact orientable embedded minimal affine Lagrangian submanifolds of a complex equiaffine space constitutes an infinite dimensional Fréchet manifold (if it is not $\emptyset$). The moduli space of all connected compact orientable metric Lagrangian embedded surfaces in an almost Kähler 4-dimensional manifold forms an infinite dimensional Fréchet manifold (if it is not $\emptyset$).

1. Introduction

R. McLean proved in [9] that special Lagrangian submanifolds near a compact special Lagrangian submanifold of a Calabi-Yau manifold form a manifold of dimension b_1, where b_1 is the first Betti number of the submanifold. Then a few papers giving generalizations to the cases where the ambient space is not a Calabi-Yau manifold but a more general type of space have been published. All those cases are, in fact, within metric geometry. The aim of this paper is to prove a similar result in the non-metric case. Moreover, we prove a global result, that is, we describe the set of all minimal affine Lagrangian embeddings of a compact manifold. It turns out that this set has a nice structure. Namely, it is an infinite dimensional Fréchet manifold modeled on the Fréchet space of all closed $(n-1)$-forms on the submanifold, where n is the complex dimension of the ambient space. The main result of this paper says that the set of all minimal affine Lagrangian embeddings of a compact manifold into an equiaffine complex space is a submanifold of the Fréchet manifold of all compact submanifolds of the complex equiaffine space. We provide a rigorous proof of this fact.

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It seems that from a differential geometry viewpoint non-metric analogues of Calabi-Yau manifolds are equiaffine complex manifolds, that is, complex manifolds equipped with a torsion-free complex connection and a non-vanishing covariant constant complex volume form. There are many very natural complex equiaffine manifolds. For instance, complex affine hyperspheres of the complex affine space $\mathbf{C}^n$ with an induced equiaffine structure obtained in a way standard in affine differential geometry (see [3]) are examples. Equiaffine structures are, in general, non-metrizable. For instance, the complex hyperspheres of $\mathbf{C}^n$ with the induced equiaffine structure are non-metrizable. In particular, they are not related to Stentzel's metric.

If N is a complex n-dimensional space with a complex structure J, then an n-dimensional real submanifold M of N is affine Lagrangian if JTM is transversal to M. Of course, if N is almost Hermitian, then Lagrangian (in the metric sense) submanifolds, for which JTM is orthogonal to TM, are affine Lagrangian and there are many affine Lagrangian submanifolds which are not metric Lagrangian even if the ambient space is almost Hermitian.

In order to discuss minimality of submanifolds a metric structure is not necessary. It is sufficient to have induced volume elements on submanifolds. Such a situation exists in case of affine Lagrangian submanifolds. In this case there does not exist (in general) any mean curvature vector but there exists the Maslov 1-form which can play, in some situations, a role similar to that played by the mean curvature vector. Note that in the general affine case we do not have any canonical duality between tangent vectors and 1-forms. The vanishing of the Maslov form implies that the submanifold is a point where a naturally defined volume functional attains its minimum for compactly supported variations. Affine Lagrangian submanifolds have a phase function. It turns out that a connected affine Lagrangian submanifold is minimal if and only if its phase function is constant. If a connected affine Lagrangian submanifold is minimal (i.e. of constant phase), then after rescaling the complex volume form in the ambient space we can assume that the constant phase function vanishes on M. Analogously to the metric case an affine Lagrangian submanifold is called special if its phase function vanishes on M. The notion of special submanifolds corresponds to the notion of calibrations. Calibrations in Riemannian geometry were introduced in the famous paper [7]. The notion can be generalized to the affine case and, like in the metric case, an affine Lagrangian submanifold is special if and only if it is calibrated by the real part of the complex volume form in the ambient space. The minimality of affine Lagrangian submanifolds is discussed in [10] and [11].

In this paper we try to assume as little as possible. In particular, we do not assume that the ambient space N is complex equiaffine but we only assume that it is almost complex and endowed with a nowhere vanishing closed complex n–form Ω, where $2n = \dim_{\mathbf{R}} N$. Then affine Lagrangian immersions $f : M \to N$, where $\dim M = n$, are those for which $f^*\Omega \neq 0$ at each point of M. If M is oriented, then Ω induces on M a unique volume element ν. We have $f^*\Omega = e^{\mathrm{i}\theta}\nu$, where θ is the phase function of f. In this paper minimal (relative to Ω) affine Lagrangian submanifolds will be those (by definition) which have constant phase.

We shall prove the following theorem.

Theorem 1.1. *Let M be a connected compact oriented n-dimensional real manifold admitting a minimal affine Lagrangian embedding into an almost complex $2n$–dimensional manifold N equipped with a nowhere-vanishing complex closed n-form.*

The set of all minimal affine Lagrangian embeddings of M into N has a structure of an infinite dimensional manifold modeled on the Fréchet vector space $\mathcal{C}^\infty(\mathcal{F}^{n-1}_{closed})$ of all smooth closed $(n-1)$–forms on M.

A precise formulation of this theorem is given in Section 4.

The Fréchet manifold in Theorem 1.1 may have many connected components. In the above theorem a manifold M is fixed. But the theorem says, in fact, about all compact (connected oriented) minimal affine Lagrangian embedded submanifolds. Non-diffeomorphic submanifolds are in different connected components.

At the end of this paper we observe that, in contrast with the metric geometry, in the affine case there exist non-smooth minimal submanifolds of smooth manifolds. Almost the same consideration as in the proof of Theorem 1.1 gives the statement saying that the set of all (metric) Lagrangian embeddings of a connected compact 2-dimensional manifold into a 4-dimensional almost Kähler manifold forms an infinite dimensional Fréchet manifold modeled on the Frëchet vector space $\mathcal{C}^\infty(\mathcal{F}^{1}_{closed})$ (if it is not $\emptyset$).

We also give a simple application of Theorem 1.1 in the case where the ambient space is the tangent bundle of a flat manifold.

2. Basic notions

Let N be a $2n$–dimensional almost complex manifold with an almost complex structure J. Let M be a connected n-dimensional manifold and $f : M \to N$ be an immersion. We say that f is affine Lagrangian (some authors call it purely real or totally real) if the bundle $Jf_*(TM)$ is transversal to $f_*(TM)$. We shall call this transversal bundle the normal bundle. The almost complex structure J gives an isomorphism between the normal bundle and the tangent bundle TM. If Ω is a nowhere vanishing complex n-form on N, then f is affine Lagrangian if and only if $f^*\Omega \neq 0$ at each point of M. If $f : M \to N$ is such a mapping that $f^*\Omega \neq 0$ at each point of M, then f is automatically an immersion.

Recall now the notion of a phase. Let $\mathbf{V}$ be an n-dimensional complex vector space with a complex volume form Ω and $\mathbf{U}$ be its n-dimensional real oriented vector subspace such that $\Omega_{|\mathbf{U}} \neq 0$. Let $X_1, ..., X_n$ be a positively oriented basis of $\mathbf{U}$. Then $\Omega(X_1, ..., X_n) = \mu e^{\mathrm{i}\theta}$, where $\mu \in \mathbf{R}^+$ and $\theta \in \mathbf{R}$. If we change the basis $X_1, ..., X_n$ to another positively oriented basis of $\mathbf{U}$, then $e^{\mathrm{i}\theta}$ remains unchanged. θ is called the phase or the angle of the subspace $\mathbf{U}$.

Assume N is endowed with a nowhere vanishing complex volume form Ω and M is oriented. For an affine Lagrangian immersion $f : M \longrightarrow N$, at each point x of M we have the phase θ_x of the tangent vector subspace $f_*(T_xM)$ of $T_{f(x)}N$. The phase function $x \longrightarrow \theta_x$ is multi-valued. In general, if we want to have the phase function to be a smooth function, it is defined only locally. For each point $x \in M$ there is a smooth phase function of f defined around x. The constancy of the phase function is a well defined global notion, that is, if θ is locally constant, then it can be chosen globally constant.

Recall few facts concerning the situation where the ambient space is complex equiaffine or, in the metric case, Calabi-Yau. A Lagrangian submanifold (affine or metric) is minimal if and only if it is volume minimizing for compactly supported variations. This is equivalent to the fact that the Maslov form vanishes. Moreover,

Lagrangian submanifolds (affine or metric) are minimal if and only if they have constant phase.

In this paper, where, in general, we do not assume that the ambient space is complex equiaffine, we shall say (by definition) that an affine Lagrangian submanifold f is minimal if and only if its phase is constant. As usual, if the phase constantly vanishes on M, then the submanifold will be called special. If the phase θ is constant, then we can rescale Ω in the ambient space by multiplying it by $\mathrm{e}^{-\mathrm{i}\theta}$ and after this change the given immersion becomes special. But if we have a family of minimal affine Lagrangian immersions of M into N, and we adjust the complex volume form Ω to one member of the family then, in general, the rest of the family remain only minimal.

For an oriented affine Lagrangian immersion $f : M \to N$ we have the induced volume form ν on M defined by the condition $\nu(X_1, ..., X_n) = |\omega(X_1, ..., X_n)|$, where $X_1, ..., X_n$ is a positively oriented basis of T_xM, $x \in M$ and $\omega = f^*\Omega$. The form ω is a real complex–valued n–form on M. We have

$$\omega = \mathrm{e}^{\mathrm{i}\theta}\nu,$$

where θ is the phase function. Note that by multiplying Ω by $\mathrm{e}^{\mathrm{i}\alpha}$ for any $\alpha \in \mathbf{R}$ we do not change the induced volume form on M. Decompose ω into the real and imaginary parts: $\omega = \omega_1 + \mathrm{i}\omega_2$, where $\omega_1 = \cos\theta\,\nu$, $\omega_2 = \sin\theta\,\nu$. If $W \in \mathcal{X}(M)$, then, since Ω is complex, we have

$$f^*(\iota_{(Jf_*W)}\Omega) = -\iota_W\omega_2 + \mathrm{i}\iota_W\omega_1,$$

where ι stands for the interior product operator. Hence, if f is special (i.e. $\nu = \omega_1$) we get

$$f^*(\iota_{(Jf_*W)}\mathrm{Im}\,\Omega) = \iota_W\nu. \tag{1}$$

Assume now that f_t, $|t| < \varepsilon$, is a smooth variation of f. Denote by $\mathcal{V}(t, x)$ its variation vector field. Assume it is normal to f at $t = 0$. Then $V := \mathcal{V}_{|\{0\}\times M}$ is equal to Jf_*W for some $W \in \mathcal{X}(M)$. If f is special and Ω is closed, then using formula (1) and Proposition (I.b.5) from [5], we obtain

$$\frac{d}{dt}\,(f_t^*\mathrm{Im}\,\Omega)_{|t=0} = d(\iota_W\nu). \tag{2}$$

This formula is also directly computed in [11], but there the form Ω is assumed to be parallel relative to a torsion-free complex connection.

We shall now give a justification of the term "minimal" adopted in this paper. Assume M is compact. If affine Lagrangian immersions $f, \tilde{f} : M \to N$ are cohomologous (in particular, if they are homotopic), then the cohomology class of ω_i is equal to the cohomology class of $\tilde{\omega}_i$, for $i = 1, 2$, where $\tilde{\omega}_1 = \cos\tilde{\theta}\,\tilde{\nu}$, $\tilde{\omega}_2 = \sin\tilde{\theta}\,\tilde{\nu}$ are the real and imaginary parts of $\tilde{\omega} = \tilde{f}^*\Omega$ and $\tilde{\theta}$, $\tilde{\nu}$ are the phase and the induced volume element for $\tilde{f}$. Assume that f is special. Then $\omega_1 = \nu$ and consequently

$$\int_M \nu = \int_M \omega_1 = \int_M \tilde{\omega}_1 = \int_M \cos\tilde{\theta}\,\tilde{\nu} \le \int_M \tilde{\nu},$$

which means that with the definition of minimality we adopted in this paper compact special (and consequently minimal) affine Lagrangian submanifolds are volume minimizing in their respective cohomology classes.

Assume additionally that $\tilde{f}$ is minimal with the constant phase $\tilde{\theta}$. We have

$$0 = \int_M \omega_2 = \int_M \tilde{\omega}_2 = \int_M \sin\tilde{\theta}\,\tilde{\nu} = \sin\tilde{\theta}\int_M \tilde{\nu},$$

which means that $\tilde{\omega}_2 = 0$, that is, $\tilde{f}$ is also special.

If f is minimal (special), then for any diffeomorphism φ of M $f \circ \varphi$ is minimal (special).

3. Moduli spaces of compact embedded submanifolds

Assume first that M and N are arbitrary manifolds such that $\dim M \leq \dim N$. Assume moreover that M is connected compact and it admits an embedding into N. Denote by $\mathcal{C}^\infty_{emb}(M, N)$ the set of all embeddings from M into N. This is a well known topological space forming an open subset (in the $\mathcal{C}^1$ topology) of $\mathcal{C}^\infty(M, N)$.

Denote by $\mathcal{M}$ the space $\mathcal{C}^\infty_{emb}(M, N)_{/Diff^\infty(M)}$ with the quotient topology. The equivalence class of $f \in \mathcal{C}^\infty_{emb}(M, N)$ will be denoted by $[f]$. For $f, g \in \mathcal{C}^\infty_{emb}(M, N)$ we have that $f \sim g$ if and only if the images of f and g are equal in N.

We shall now introduce a structure of an infinite dimensional manifold (modeled on Fréchet spaces) on $\mathcal{M}$. It is certainly well known but we have not found suitable references and moreover we need the construction. We use the notion of a manifold modeled on Fréchet vector spaces given in [6]. We denote the Fréchet space of all $\mathcal{C}^\infty$ sections of a vector bundle $E \to M$ by $\mathcal{C}^\infty(M \leftarrow E)$. Analogously the Banach spaces of all $\mathcal{C}^k$ sections of a vector bundle $E \to M$ will be denoted by $\mathcal{C}^k(M \leftarrow E)$.

The basic tool in the construction are tubular mappings. We use the following setting of this notion. Assume that $\mathcal{N}_f$ is any smooth transversal bundle for an embedding $f : M \to N$. Having any connection on N we have the exponential mapping *exp* given by the connection. No relation between the connection and the transversal bundle is needed. From the theory of connections one knows that there is an open neighborhood $\mathcal{U}$ of the zero-section in the total space $\mathcal{N}_f$ and an open neighborhood $\mathcal{T}$ of $f(M)$ in N such that $exp_{|\mathcal{U}} : \mathcal{U} \to \mathcal{T}$ is a diffeomorphism, $exp_{|M} = \mathrm{id}_M$ and the differential $exp_* : T_{0_x}(\mathcal{N}_f) = f_*(T_xM) \oplus (\mathcal{N}_f)_x \to T_xN$ of *exp* at 0 is the identity for each point x of M. The mapping $exp_{|\mathcal{U}}$ is a tubular mapping. In order to reduce a play with neighborhoods we shall use the following lemma, which allows to have the whole total space $\mathcal{N}_f$ as the domain of a tubular mapping. In what follows $\mathcal{N}_f$ will denote either the transversal vector bundle or its total space depending on the context.

Lemma 3.1. *Let $E \longrightarrow M$ be a Riemannian vector bundle and $\mathcal{U}_\varepsilon$ be the neighbourhood of the zero section of E given as follows*

$$\mathcal{U}_\varepsilon = \{v \in E;\ \mid v \mid < \varepsilon\},$$

where $\mid\ \mid$ is the norm on fibers of E determined by the Riemannian structure. There is a fiber-respecting diffeomorphism $\sigma : E \longrightarrow \mathcal{U}_\varepsilon$ which is the identity on $\mathcal{U}_{\varepsilon/2}$.

Proof. Let $\psi : [0, \infty) \to \mathbf{R}$ be a smooth function such that $\psi(t) = t$ for $t \leq \varepsilon/2$, $\psi(t) \leq \varepsilon$ for $t > \varepsilon/2$ and $\psi(t) \to \varepsilon$ for $t \to \infty$. Then the function $\Upsilon(t) = (1/t)\psi(t)$ is also a smooth function on $[0, \infty)$. The mapping $\sigma : E \to \mathcal{U}_\varepsilon$ given by

$$\sigma(v) = \Upsilon(\mid v \mid)v$$

satisfies the required conditions. □

We now endow the bundle $\mathcal{N}_f$ with any Riemannian metric. Since M is compact, there is $\varepsilon > 0$ such that $\mathcal{U}_\varepsilon \subset \mathcal{U}$. We use Lemma 3.1 for $\mathcal{U}_\varepsilon$ and take the tubular mapping $\mathcal{E}_f = exp \circ \sigma$. The tubular neighborhood $\mathcal{E}_f(\mathcal{N}_f)$ of $f(M)$ will be denoted by $\mathcal{T}_f$.

The set $\mathcal{C}^\infty_{emb}(M, \mathcal{T}_f)$ is open in $\mathcal{C}^\infty_{emb}(M, N)$ (in the $\mathcal{C}^0$-topology). Consider the mapping

$$\Psi : \mathcal{C}^\infty_{emb}(M, \mathcal{T}_f) \ni h \longrightarrow \Pi_f \circ \mathcal{E}_f^{-1} \circ h \in \mathcal{C}^\infty(M, M), \tag{3}$$

where $\Pi_f : \mathcal{N}_f \longrightarrow M$ is the natural projection. The mapping is continuous and the set $\mathcal{D}iff^\infty(M)$ is open in $\mathcal{C}^\infty(M, M)$ (in the $\mathcal{C}^1$-topology). Thus the set

$$\mathcal{U}^1_f = \Psi^{-1}(\mathcal{D}iff^\infty(M)) = \{h \in \mathcal{C}^\infty(M, \mathcal{T}_f);\ \Pi_f \circ \mathcal{E}_f^{-1} \circ h \in \mathcal{D}iff^\infty(M)\} \tag{4}$$

is open in $\mathcal{C}^\infty_{emb}(M, N)$ in the $\mathcal{C}^1$ topology. Observe that $h \in \mathcal{U}^1_f$ if and only if there is a section $V \in \mathcal{C}^\infty(M \longleftarrow \mathcal{N}_f)$ and $\varphi \in \mathcal{D}iff^\infty(M)$ such that

$$\mathcal{E}_f \circ V = h \circ \varphi. \tag{5}$$

The set $\mathcal{U}^1_f$ has the following properties:

1) If $h \in \mathcal{U}^1_f$ and $\varphi \in \mathcal{D}iff^\infty(M)$, then $h \circ \varphi \in \mathcal{U}^1_f$.

2) For every $\varphi \in \mathcal{D}iff^\infty(M)$ we have $\mathcal{U}^1_{f\circ\varphi} = \mathcal{U}^1_f$.

Take the neighborhood $\mathcal{U}^1_{[f]} = \{[h] \in \mathcal{M};\ h \in \mathcal{U}^1_f\}$ of $[f]$ in $\mathcal{M}$. Observe that the elements of $\mathcal{U}^1_{[f]}$ can be parametrized simultaneously. Namely, we have

Lemma 3.2. *Let $\xi_0 \in \mathcal{U}^1_{[f]}$ and $h_0 \in \mathcal{C}^\infty_{emb}(M, N)$ be its fixed parametrization. For each $\xi \in \mathcal{U}^1_{[f]}$ there is a unique parametrization $h_\xi \in \mathcal{C}^\infty_{emb}(M, N)$ of ξ such that*

$$\Pi_f \circ \mathcal{E}_f^{-1} \circ h_0 = \Pi_f \circ \mathcal{E}_f^{-1} \circ h_\xi$$

Proof. We first reparametrize f in such a way that after the reparametrization

$$\Pi_f \circ \mathcal{E}_f^{-1} \circ h_0 = \mathrm{id}_M.$$

Assume that f is already parametrized in this way. For every $h \in \mathcal{U}^1_f$ the mapping $\varphi^{-1} = \Pi_f \circ \mathcal{E}_f^{-1} \circ h$ is a diffeomorphism and it is sufficient to replace h representing $[h]$ by $h \circ \varphi$. The uniqueness is obvious. □

By the above lemma we see that $\mathcal{U}^1_{[f]}$ can be identified with the set

$$\mathcal{U}_{[f]} = \{h \in \mathcal{C}^\infty_{emb}(M, \mathcal{T}_f);\ \Pi_f \circ \mathcal{E}_f^{-1} \circ h = \mathrm{id}_M\}. \tag{6}$$

We now define the bijection

$$u_{[f]} : \mathcal{U}_{[f]} \longrightarrow \mathbf{C}^\infty(M \leftarrow \mathcal{N}_f)$$

as follows:

$$u_{[f]}(h) \longrightarrow \mathcal{E}_f^{-1} \circ h. \tag{7}$$

We see that

$$u^{-1}_{[f]}(V) = \mathcal{E}_f \circ V$$

and $\mathcal{E}_f \circ V$ has values in $\mathcal{T}_f$. If U is an open subset of $\mathcal{T}_f$, then

$$u_{[f]}(\{h \in \mathcal{U}_{[f]}; h(M) \subset U\}) = \{V \in \mathcal{C}^\infty(M \leftarrow \mathcal{N}_f);\ V(M) \subset \mathcal{E}_f^{-1}(U)\}$$

and hence is open in $\mathcal{C}^\infty(M \leftarrow \mathcal{N}_f)$.

Assume now that $f, g \in \mathcal{C}^{\infty}_{emb}(M,N)$ and $\mathcal{U}_{[f]} \cap \mathcal{U}_{[g]} \neq \emptyset$. Take $\xi_0 \in \mathcal{U}_{[f]} \cap \mathcal{U}_{[g]}$ and fix its parametrization h_0. Reparametrize f and g as in Lemma 3.2 adjusting the parametrizations to h_0. Then

$$\begin{aligned}\mathcal{U}_{[f]} \cap \mathcal{U}_{[g]} &= \{h \in \mathcal{C}^{\infty}_{emb}(M, \mathcal{T}_f \cap \mathcal{T}_g);\ \Pi_f \circ \mathcal{E}_f^{-1} \circ h = \mathrm{id}_M,\ \Pi_g \circ \mathcal{E}_g^{-1} \circ h = \mathrm{id}_M\} \\ &= \{\mathcal{E}_f \circ V;\ V \in \mathcal{C}^{\infty}(M \leftarrow \mathcal{N}_f);\ V(M) \subset \mathcal{E}_f^{-1}(\mathcal{T}_f \cap \mathcal{T}_g)\}\end{aligned}$$

and consequently

$$u_{[f]}(\mathcal{U}_{[f]} \cap \mathcal{U}_{[g]}) = \{\ V \in \mathcal{C}^{\infty}(M \leftarrow \mathcal{N}_f);\ V(M) \subset \mathcal{E}_f^{-1}(\mathcal{T}_f \cap \mathcal{T}_g)\}$$

The mapping $\mathcal{E}_g^{-1} \circ \mathcal{E}_f : \mathcal{E}_f^{-1}(\mathcal{T}_f \cap \mathcal{T}_g) \to \mathcal{E}_g^{-1}(\mathcal{T}_f \cap \mathcal{T}_g)$ is smooth and fiber respecting (because of specially chosen parametrizations f and g). It is known, [6], that the set $u_{[f]}(\mathcal{U}_{[f]} \cap \mathcal{U}_{[g]})$ is open in the Fréchet space $\mathcal{C}^{\infty}(M \leftarrow \mathcal{N}_f)$ and the mapping

$$u_{[g][f]} : u_{[f]}(\mathcal{U}_{[f]} \cap \mathcal{U}_{[g]}) \ni V \to \mathcal{E}_g^{-1} \circ \mathcal{E}_f \circ V \in u_{[g]}(\mathcal{U}_{[f]} \cap \mathcal{U}_{[g]})$$

is smooth. For the same reason the set $u_{[g]}(\mathcal{U}_{[f]} \cap \mathcal{U}_{[f]})$ is open and the mapping $u_{[f][g]}$ is smooth.

We have built a smooth atlas on $\mathcal{M}$. Hence we have

Theorem 3.3. *Let M be a connected compact manifold admitting an embedding in a manifold N. Then $\mathcal{M}$ is an infinite dimensional manifold modeled on the Fréchet vector spaces $\mathcal{C}^{\infty}(M \leftarrow \mathcal{N}_f)$ for $f \in \mathcal{C}^{\infty}_{emb}(M,N)$, where $\mathcal{N}_f$ is any bundle transversal to f.*

In the theorem $\mathcal{N}_f$ can be replaced by any bundle isomorphic (over the identity on M) to the transversal bundle $\mathcal{N}_f$.

In what follows the Fréchet space of all smooth r-forms on M will be denoted by $\mathcal{C}^{\infty}(\mathcal{F}^r)$. The Banach space of r-forms of class $\mathcal{C}^k$, $k \in \mathbf{N}$, will be denoted by $\mathcal{C}^k(\mathcal{F}^r)$.

Assume now additionally that N is a $2n$-dimensional manifold with an almost complex structure J and M is n-dimensional orientable with a fixed volume form ν. Having the volume element ν, we have an isomorphic correspondence between tangent vectors and $(n-1)$-forms. It is given by the interior multiplication

$$T_x M \ni W \longrightarrow \iota_W \nu \in \Lambda^{n-1}(T_x M)^*,$$

for $x \in M$. If $f : M \to N$ is affine Lagrangian, then by composing this isomorphism with the isomorphism determined by J between the tangent bundle TM and the normal bundle $\mathcal{N}_f$ we get an isomorphism, say ρ, of vector bundles

$$\rho : \Lambda^{n-1} TM^* \longrightarrow \mathcal{N}_f. \tag{8}$$

The isomorphism gives a smooth isomorphism (linear smooth diffeomorphism) $\wp$ between Fréchet vector spaces $\mathcal{C}^{\infty}(\mathcal{F}^{n-1})$ and $\mathcal{C}^{\infty}(M \leftarrow \mathcal{N}_f)$ given by $\wp(\gamma) = \rho \circ \gamma$.

We now have

Theorem 3.4. *Let M be a connected compact orientable n-dimensional real manifold admitting an affine Lagrangian embedding into a $2n$-dimensional almost complex manifold N. The set $\mathcal{M}aL = \{[f] \in \mathcal{M};\ f \text{ is } affine \text{ Lagrangian}\}$ is an infinite dimensional manifold modeled on the Fréchet vector space $\mathbf{C}^{\infty}(\mathcal{F}^{n-1})$.*

Proof. For each $y \in N$ there is an open neighborhood U_y of y in N and a smooth complex n-form Ω_y on N such that $\Omega_y \neq 0$ at each point of U_y. Let $U_{y_1}, ..., U_{y_l}$ cover $f(M)$. Set $\tilde{\Theta}_j = {\mathcal{E}_f}^* \Omega_{y_j}$. Consider the mapping

$$\mathcal{C}^1(M \leftarrow \mathcal{N}_f) \ni V \to (V^*\tilde{\Theta}_1, ..., V^*\tilde{\Theta}_l) \in (C^0(\mathcal{F}(\mathbf{C})))^l. \tag{9}$$

where $\mathcal{C}^0(\mathcal{F}(\mathbf{C}))$ stands for the space of all real complex-valued n-forms on M of class $\mathcal{C}^0$. It is known, see Theorem 2.2.15 from [1], that this mapping is continuous between Banach spaces. Hence

$$\tilde{\mathcal{U}} = \{V \in \mathcal{C}^\infty(M \leftarrow \mathcal{N}_f);\ ((V^*\tilde{\Theta}_1)_x, ..., (V^*\tilde{\Theta}_l)_x) \neq 0\ \forall x \in M\}$$

is open in $\mathcal{C}^\infty(M \leftarrow \mathcal{N}_f)$. It is clear that $[h] \in \mathcal{U}^1_{[f]}$ is affine Lagrangian if and only $u_{[f]}([h]) \in \tilde{\mathcal{U}}$. We now compose $u_{[f]}$ with the isomorphism $\wp^{-1}$, where $\wp$ is determined by any fixed volume form on M. $\square$

In the above atlas we can compose a chart $u_{[f]}$ with a bijective mapping, say ϕ, sending an open neighborhood of 0 in $\mathbf{C}^\infty(\mathcal{F}^{n-1})$ onto an open neighborhood of 0 in $\mathbf{C}^\infty(\mathcal{F}^{n-1})$ and such that ϕ and ϕ^{-1} are smooth in the sense of the theory of Fréchet vector spaces. This does not change the differentiable structure on $\mathcal{M}aL$. We shall use this possibility in the next section.

4. The moduli space of minimal submanifolds

The precise formulation of Theorem 1.1 is the following

Theorem 4.1. *Let N be a $2n$-dimensional almost complex manifold equipped with a smooth nowhere-vanishing closed complex n-form Ω. Let M be a connected compact oriented n-dimensional real manifold admitting a minimal (relative to Ω) affine Lagrangian embedding into N. Then the set*

$$\mathcal{M}maL = \{[f] \in \mathcal{M}aL;\ f \text{ is minimal}\}$$

is an infinite dimensional manifold modeled on the Fréchet vector space $\mathbf{C}^\infty(\mathcal{F}^{n-1}_{closed})$. It is a submanifold of $\mathcal{M}aL$.

Proof. We shall improve the charts obtained in Theorem 3.4 in such a way that the set $\mathcal{M}maL$ will get a structure of a submanifold of $\mathcal{M}aL$ in the sense of the theory of Fréchet manifolds. Let $f : M \to N$ be a given minimal affine Lagrangian embedding. By rescaling Ω in the ambient space we make f special. We have the normal bundle $\mathcal{N}_f = Jf_*(TM)$. Fix a tubular mapping $\mathcal{E}_f : \mathcal{N}_f \to \mathcal{T}_f$. For each section $V \in \mathcal{C}^\infty(M \leftarrow \mathcal{N}_f)$ we have the embedding $f_V = \mathcal{E}_f \circ V$. In general, f_V is neither special nor minimal nor even affine Lagrangian. Consider the mapping

$$\tilde{P} : \mathcal{C}^\infty(M \leftarrow \mathcal{N}_f) \ni V \to \tilde{P}(V) = f_V^*(\mathrm{Im}\ \Omega) \in \mathcal{C}^\infty(\mathcal{F}^n).$$

Of course $\tilde{P}(0) = 0$. For a section $V \in \mathcal{C}^\infty(M \leftarrow \mathcal{N}_f)$ take the variation $f_t = f_{tV}$. The section V is the variation vector field for f_t at 0. Using now formula (2) one sees that the linearization $L_0\tilde{P}$ of $\tilde{P}$ at 0 is given by the formula

$$L_0\tilde{P}(V) = d(\iota_W \nu), \tag{10}$$

where $V = Jf_*W$ and ν is the volume form on M induced by f.

Since for each $V \in \mathcal{C}^\infty(M \leftarrow \mathcal{N}_f)$ the embedding f_V is homotopic to f, we have that $\tilde{P}$ has values in $\mathcal{C}^\infty(\mathcal{F}^n_{exact})$. Moreover, as it was observed in Section 2, if f_V is minimal affine Lagrangian, then it is automatically special.

We shall now use the isomorphism ρ given by (8). If $\gamma \in \mathcal{C}^\infty(\mathcal{F}^{n-1})$ and $V = \rho\circ\gamma$, then $\gamma = \iota_W \nu$, where $V = Jf_*W$. We now have the mapping

$$P : \mathcal{C}^\infty(\mathcal{F}^{n-1}) \longrightarrow \mathcal{C}^\infty(\mathcal{F}^n_{exact})$$

defined as follows:

$$P(\gamma) = \tilde{P}(\rho \circ \gamma). \tag{11}$$

The mapping P can be also expressed as follows. If we set $\Theta = (\mathcal{E}_f \circ \rho)^* \mathrm{Im}\,\Omega$, then Θ is a closed n-form on the total space of $\Lambda^{n-1}TM^*$. We have $P(\gamma) = \gamma^*\Theta$ for any $(n-1)$ - form γ. Obviously $P(0) = 0$. Moreover, $P(\gamma) = 0$ if and only if f_V, where $V = \rho \circ \gamma$, is special (if f_V is affine Lagrangian).

We shall now regard P as a differential operator. It is smooth, of order 1, non-linear and, by (10), the linearization L_0P of P at 0 is given by

$$L_0P = d \tag{12}$$

We shall now fix an arbitrary positive definite metric tensor field on M. The metric is only a tool here and has no relation with the affine geometric structure considered in this paper. Denote by δ the codifferential operator determined by the metric. Denote by $\mathcal{C}^{k,a}(\mathcal{F}^r)$ the Hölder-Banach space of all r-forms on M of class $\mathcal{C}^{k,a}$, where $k \in \mathbf{N}$ and a is a real number from $(0, 1)$.

We extend the action of the operators P, d, δ to the action on the forms of class $\mathcal{C}^{k,a}$. The extensions will be denoted by the same letters. In particular, after extending, P becomes a $\mathcal{C}^\infty$ mapping between Banach spaces, see [1] p. 34,

$$P : \mathcal{C}^{k,a}(\mathcal{F}^{n-1}) \longrightarrow \mathcal{C}^{k-1,a}({\mathcal{F}^n}_{exact}) \tag{13}$$

for each $k = 1, 2, ...$ As in the proof of Theorem 3.4 one sees that there is an open neighborhood, say $\mathcal{W}$, of 0 in $\mathcal{C}^{1,a}(\mathcal{F}^{n-1})$ such that f_V are affine Lagrangian for $V = \rho \circ \gamma$ and $\gamma \in \mathcal{W}$. From now on all neighborhoods of 0 in $\mathcal{C}^{1,a}(\mathcal{F}^{n-1})$ will be contained in $\mathcal{W}$. Moreover, all neighborhoods will be assumed open.

Consider now $P : \mathcal{C}^{1,a}(\mathcal{F}^{n-1}) \longrightarrow \mathcal{C}^{0,a}({\mathcal{F}^n}_{exact})$ as a mapping between Banach spaces. The mapping L_0P is a surjection. Moreover $\ker L_0P = \ker d$. Denote the Banach space $\mathcal{C}^{1,a}(\mathcal{F}^{n-1}_{closed}) = \ker d \subset \mathcal{C}^{1,a}(\mathcal{F}^{n-1})$ by X. The space $\delta(\mathcal{C}^{2,a}(\mathcal{F}^n))$ is a closed complement to $\ker d$. Denote this Banach space by Y. Using the implicit mapping theorem for Banach spaces one gets that there is an open neighborhood A of 0 in X and an open neighborhood B of 0 in Y and a unique smooth mapping $G : A \longrightarrow B$ such that

$$(A + B) \cap P^{-1}(0) = \{\alpha + G(\alpha);\ \alpha \in A\}.$$

We shall now observe that if α is of class $\mathcal{C}^{k,a}$, where $k \geq 2$ or of class $\mathcal{C}^\infty$, then so is $G(\alpha)$, for α from some neighborhood of 0 in X. In Riemannian geometry special submanifolds as minimal ones are automatically $\mathcal{C}^\infty$ (after possible reparametrization), but in the affine case we do not have such a statement and we have to prove that α of class $\mathcal{C}^\infty$ give rise to a smooth embedding.

For an $(n-1)$–form γ we define a differential operator P_γ of the second order from the vector bundle Λ^nTM^* into itself by the formula

$$P_\gamma(\beta) = P(\gamma + \delta\beta) \tag{14}$$

for an n-form β. Since $d\beta = 0$, the linearization of P_0 at 0 is the Laplace operator. We also have

$$L_\beta P_\gamma = L_{\gamma+\delta\beta} P \circ \delta. \tag{15}$$

Hence, if γ is of class $\mathcal{C}^{k,a}$ and β is of class $\mathcal{C}^{k+1,a}$, then the linear differential operator $L_\beta P_\gamma$ is of class $\mathcal{C}^{k-1,a}$.

We have the following smooth mapping between Banach spaces

$$\mathcal{C}^{1,a}(\mathcal{F}^{n-1}) \times \mathcal{C}^{2,a}(\mathcal{F}^n) \ni (\gamma, \beta) \longrightarrow P_\gamma(\beta) \in \mathcal{C}^{0,a}(\mathcal{F}^n)$$

and the continuous mapping

$$\Phi : STM^* \times \mathcal{C}^{1,a}(\mathcal{F}^{n-1}) \times \mathcal{C}^{2,a}(\mathcal{F}^n) \ni (\xi, \alpha, \beta) \longrightarrow \det\, \sigma_\xi(L_\beta P_\alpha) \in \mathbf{R}, \tag{16}$$

where STM^* stands for the total space of the unit spheres bundle in TM^* and σ_ξ denotes the principal symbol of a differential operator. Since STM^* is compact and $\Phi(\xi, 0, 0) \neq 0$ for every $\xi \in STM^*$, we obtain the following

Lemma 4.2. *There is a neighborhood $\mathcal{U}_0$ of 0 in $\mathcal{C}^{1,a}(\mathcal{F}^{n-1})$ and a neighborhood $\mathcal{V}_0$ of 0 in $\mathcal{C}^{2,a}(\mathcal{F}^n)$ such that for each $\gamma \in \mathcal{U}_0$ and $\beta \in \mathcal{V}_0$ the differential operator $L_\beta P_\gamma$ is elliptic.*

From the theory of elliptic differential operators applied to $d + \delta$ we know that the codifferential (after restricting) is a linear homemorphism of Banach spaces

$$\delta : \mathcal{C}^{2,a}(\mathcal{F}^n_{exact}) \longrightarrow Y = \delta(\mathcal{C}^{2,a}(\mathcal{F}^n)).$$

Take the neighborhood of 0 in X given by $\mathcal{U}_1 = G^{-1}(\delta(\mathcal{V}_0 \cap \mathcal{C}^{2,a}(\mathcal{F}^n_{exact})) \cap \mathcal{U}_0$. Let $\alpha \in \mathcal{U}_1$. Then $G(\alpha)$ exists and there exists $\beta \in \mathcal{V}_0$ such that $G(\alpha) = \delta\beta$. Moreover $P_\alpha(\beta) = 0$ and $L_\beta P_\alpha$ is elliptic. Take now any $k \geq 2$ and $\alpha \in \mathcal{U}_1 \cap \mathcal{C}^{k,a}(\mathcal{F}^{n-1})$. Then the differential operator P_α is of class $\mathcal{C}^{k-1,a}$. For $G(\alpha)$ we have $\beta \in \mathcal{V}_0$ of class $\mathcal{C}^2$ such that $G(\alpha) = \delta\beta$, i.e. $P_\alpha(\beta) = 0$. Hence β is an elliptic solution of the equation $P_\alpha(\beta) = 0$ and from the elliptic regularity theorem for non-linear differential operators we know that β is of class $\mathcal{C}^{k+1,a}$ and consequently $G(\alpha) = \delta\beta$ is of class $\mathcal{C}^{k,a}$. Thus if α is of class $\mathcal{C}^\infty$ then so is $G(\alpha)$. We have got

Lemma 4.3. *There is a neighborhood $\mathcal{U}_1$ of 0 in X such that for each $k \geq 1$ we have the mapping*

$$G_{|\mathcal{C}^{k,a}(\mathcal{F}^{n-1})} : \mathcal{U}_1 \cap \mathcal{C}^{k,a}(\mathcal{F}^{n-1}) \longrightarrow Y \cap \mathcal{C}^{k,a}(\mathcal{F}^{n-1}) = \delta(\mathcal{C}^{k+1,a}(\mathcal{F}^n)). \tag{17}$$

Consequently we have the mapping

$$G_{|\mathcal{C}^\infty(\mathcal{F}^{n-1})} : \mathcal{U}_1 \cap \mathcal{C}^\infty(\mathcal{F}^{n-1}) \longrightarrow Y \cap \mathcal{C}^\infty(\mathcal{F}^n) = \delta(\mathcal{C}^\infty(\mathcal{F}^n)). \tag{18}$$

We know that $G : \mathcal{U}_1 \to Y$ is smooth between Banach spaces. We shall now prove that the mappings (17), for $k = 2, \ldots$, are smooth mappings between Banach spaces, when we replace $\mathcal{U}_1$ by a (possibly) smaller neighborhood of 0 in X. It will imply that the mapping (18) is smooth as a mapping between Fréchet spaces in a sufficiently small neighborhood of 0 in $\mathcal{C}^\infty(\mathcal{F}^{n-1})$.

We have the continuous mapping

$$\mathcal{C}^{1,a}(\mathcal{F}^{n-1}) \ni \gamma \longrightarrow (L_\gamma P)_{|Y} \in \mathcal{L}(Y, Z), \tag{19}$$

where $Z = \mathcal{C}^{0,a}(\mathcal{F}^n_{exact})$ and $\mathcal{L}(Y, Z)$ stands for the Banach space of continuous linear mappings from Y to Z. We know that $(L_0 P)_{|Y} = d_{|Y} : Y \to Z$ is an isomorphism (linear and topological) between Banach spaces Y, Z. Since the set of isomorphisms is open in $\mathcal{L}(Y, Z)$, there is a neighborhood, say $\mathcal{U}_2$, of 0 in $\mathcal{C}^{1,a}(\mathcal{F}^{n-1})$

such that if $\gamma \in \mathcal{U}_2$, then $(L_\gamma P)_{|Y}$ is an isomorphism between Y and Z. Take $\gamma \in \mathcal{U}_2 \cap \mathcal{C}^{k,a}(\mathcal{F}^{n-1})$, We have the mapping

$$(L_\gamma P)_{|Y_k} : Y_k \longrightarrow \mathcal{C}^{k-1,a}(\mathcal{F}^n_{exact}) = Z \cap \mathcal{C}^{k-1,a}(\mathcal{F}^n), \tag{20}$$

where $Y_k = Y \cap \mathcal{C}^{k,a}(\mathcal{F}^{n-1}) = \delta(\mathcal{C}^{k+1,a}(\mathcal{F}^n))$. As a restriction of the injection $L_\gamma P : Y \to Z$, it is injective. Since $P_{|\mathcal{C}^{k,a}(\mathcal{F}^{n-1})} : \mathcal{C}^{k,a}(\mathcal{F}^{n-1}) \longrightarrow \mathcal{C}^{k-1,a}(\mathcal{F}^n_{exact})$ is smooth between Banach spaces, we have that

$$L_\gamma \left(P_{|\mathcal{C}^{k,a}(\mathcal{F}^{n-1})}\right) : \mathcal{C}^{k,a}(\mathcal{F}^{n-1}) \longrightarrow \mathcal{C}^{k-1,a}(\mathcal{F}^n_{exact})$$

is continuous. Hence

$$\left(L_\gamma \left(P_{|\mathcal{C}^{k,a}(\mathcal{F}^{n-1})}\right)\right)_{|Y_k} : Y_k \longrightarrow \mathcal{C}^{k-1,a}(\mathcal{F}^n_{exact})$$

is continuous. On the other hand

$$L_\gamma \left(P_{|\mathcal{C}^{k,a}(\mathcal{F}^{n-1})}\right) = (L_\gamma P)_{|\mathcal{C}^{k,a}(\mathcal{F}^{n-1})}.$$

Thus the mapping given by (20) is a continuous linear monomorphism. We shall now show that it is surjective for $\gamma \in \mathcal{U}_2 \cap \mathcal{U}_0 \cap \mathcal{C}^{k,a}(\mathcal{F}^{n-1})$. Let $\mu \in \mathcal{C}^{k-1,a}(\mathcal{F}^n_{exact})$. Since $\gamma \in \mathcal{U}_0$, by Lemma 4.2, we know that $L_0 P_\gamma$ is elliptic. The differential operator $L_0 P_\gamma$ is of class $\mathcal{C}^{k-1,a}$. Since $\gamma \in \mathcal{U}_2$, there is $\beta \in \mathcal{C}^{2,a}(\mathcal{F}^n)$ such that $L_\gamma P(\delta\beta) = \mu$. From the elliptic regularity theorem we know that β is of class $\mathcal{C}^{k+1,a}$, i.e. $\delta\beta$ is of class $\mathcal{C}^{k,a}$. Set $\mathcal{U}_3 = \mathcal{U}_0 \cap \mathcal{U}_2$. We have got

Lemma 4.4. *There is a neighborhood $\mathcal{U}_3$ of 0 in $\mathcal{C}^{1,a}(\mathcal{F}^{n-1})$ such that for every $\gamma \in \mathcal{U}_3 \cap \mathcal{C}^{k,a}(\mathcal{F}^{n-1})$ the mapping*

$$L_\gamma P_{|Y_k} : Y_k \longrightarrow \mathcal{C}^{k-1,a}(\mathcal{F}^n_{exact})$$

is an isomorphism (topological and linear).

Denote by $\tilde{G} : A \to \mathcal{C}^{1,a}(\mathcal{F}^{n-1})$ the mapping given by $\tilde{G}(\alpha) = \alpha + G(\alpha)$. Take $\mathcal{U}_4 = \tilde{G}^{-1}(\mathcal{U}_3) \cap \mathcal{U}_1 \subset X$. Let $\alpha_0 \in \mathcal{U}_4 \cap \mathcal{C}^{k,a}(\mathcal{F}^{n-1})$. Then $\gamma_0 = \alpha_0 + G(\alpha_0)$ is of class $\mathcal{C}^{k,a}$ (because $\alpha_0 \in \mathcal{U}_1$) and $L_{\gamma_0} P_{|Y_k} : Y_k \to \mathcal{C}^{k-1,a}(\mathcal{F}^n_{exact})$ is an isomorphism (because $\alpha_0 \in \tilde{G}^{-1}(\mathcal{U}_3)$). Denote by $\tilde{X}_k$ the Banach space $\ker L_{\gamma_0} P$. We have $P(\gamma_0) = 0$ and $\tilde{X}_k \oplus Y_k = \mathcal{C}^{k,a}(\mathcal{F}^{n-1})$. We want to prove that G is smooth around α_0 in the sense of the Banach spaces theory. Denote by $\tilde{\pi} : \mathcal{C}^{k,a}(\mathcal{F}^{n-1}) = \tilde{X}_k \oplus Y_k \longrightarrow \tilde{X}_k$ the canonical projection. It is a smooth mapping between Banach spaces. Set $\tilde{\alpha}_0 = \tilde{\pi}(\alpha_0)$. From the implicit mapping theorem we know that there is a neighborhood $\tilde{U}$ of $\tilde{\alpha}_0$ in $\tilde{X}_k$ and a smooth mapping F defined on $\tilde{U}$ such that $\{\tilde{\alpha} + F(\tilde{\alpha}); \tilde{\alpha} \in \tilde{U}\} \subset P^{-1}(0)$. In a neighborhood of α_0 we have

$$G(\alpha) = \tilde{\pi}(\alpha) + F(\tilde{\pi}(\alpha)) - \alpha.$$

Hence in a neighborhood of α_0 the mapping G is smooth and consequently it is smooth in $\mathcal{U}_4 \cap \mathcal{C}^{k,a}(\mathcal{F}^{n-1})$. It follows that $G_{|\mathcal{U}_4 \cap \mathcal{C}^\infty(\mathcal{F}^{n-1})}$ is smooth in the sense of the theory of Fréchet spaces.The projections in the Hodge decomposition $\mathcal{C}^\infty(\mathcal{F}^{n-1}) = \mathcal{C}^\infty(\mathcal{F}^{n-1}_{closed}) \oplus \delta(\mathcal{C}^\infty(\mathcal{F}^n))$ are smooth mappings of Fréchet spaces. Denote by $p : \mathcal{C}^\infty(\mathcal{F}^{n-1}) \to \mathcal{C}^\infty(\mathcal{F}^{n-1}_{closed})$ the projection. Set $\mathfrak{U} = (\mathcal{U}_4 \cap \mathcal{C}^\infty(\mathcal{F}^{n-1})) \oplus (\delta(\mathcal{C}^\infty(F^n))$. Consider the mapping $\phi : \mathfrak{U} \to \mathfrak{U}$ defined as $\phi(z) = z - (G \circ p)(z)$. It is a bijection and its converse is given by $\phi^{-1}(z) = z + (G \circ p)(z)$. Both mappings ϕ and ϕ^{-1} are smooth in the sense of the theory of Fréchet vector spaces. We now compose the chart obtained in the proof of Theorem 3.4 with ϕ. Since

$$\phi(\{\gamma = \alpha + G(\alpha);\ \alpha \in \mathcal{U}_4 \cap \mathcal{C}^\infty(\mathcal{F}^{n-1})\}) = \mathcal{U}_4 \cap \mathcal{C}^\infty(\mathcal{F}^{n-1})$$

is an open subset of the closed subspace $\mathcal{C}^\infty(\mathcal{F}^\infty_{closed})$ of $\mathcal{C}^\infty(\mathcal{F}^\infty)$, we have that the set $\mathcal{M}maL$ is a submanifold of $\mathcal{M}aL$. The proof is completed.

Remark 4.5. We shall now observe that there exist minimal affine Lagrangian submanifolds which are not smooth. We refer to Section 3 for notation. Having a smooth special affine Lagrangian embedding $f : M \to N$ we have the mapping

$$\Phi_k : \mathcal{C}^k_{emb}(M, \mathcal{T}_f) \ni h \longrightarrow \Pi_f \circ \mathcal{E}_f^{-1} \circ h \in \mathcal{C}^k(M, M).$$

The set $Diff^k(M)$ is open in $\mathcal{C}^k(M, M)$ and the set

$$\mathcal{U}_{k,f} = \{h \in \mathcal{C}^k_{emb}(M, \mathcal{T}_f);\ \exists V \in \mathcal{C}^k(M \leftarrow \mathcal{N}_f) : \mathcal{E}_f \circ V = h\}$$

can be regarded as an open neighbourhood of $[f]$ in $\mathcal{M}^k = \mathcal{C}^k(M, \mathcal{T}_f)_{/Diff^k(M)}$. We have the bijection

$$\mathcal{C}^k(M, \leftarrow \mathcal{N}_f) \ni V \longrightarrow \mathcal{E}_f \circ V \in \mathcal{U}_{k,f}. \tag{21}$$

In order to study minimal affine Lagrangian submanifolds of complex equiaffine spaces like in [10], [11] or in Section 1 of this paper it suffices that the immersions or embeddings under consideration are of class $\mathcal{C}^2$. Also in the proof of Theorem 4.1 the class $\mathcal{C}^2$ is sufficient, that is, if α is of class $\mathcal{C}^k$, where $k \geq 2$, then $G(\alpha)$ is of class $\mathcal{C}^k$. Since $\mathcal{C}^k(\mathcal{F}^{n-1}_{closed}) \neq \mathcal{C}^\infty(\mathcal{F}^{n-1}_{closed})$, it is clear by the proof of Theorem 4.1 that there exist non-smooth minimal affine Lagrangian embeddings of class $\mathcal{C}^k$, for $k \geq 2$.

Example 4.6. Let M be an n-dimensional real manifold equipped with a torsion-free linear connection ∇. The tangent bundle to the tangent bundle TTM admits a decomposition into a direct sum of the vertical bundle (tangent to the fibers of TM) and the horizontal bundle (depending on the connection). The vertical lift of $X \in T_xM$ to T_ZTM for $Z \in T_xM$ will be denoted by X^v_Z. Analogously the horizontal lift will be denoted by X^h_Z. The following formulas for the lifts of vector fields $X, Y \in \mathcal{X}(M)$ are known, see [4],

$$\begin{aligned} &[X^v, Y^v] = 0,\\ &[X^h, Y^v] = (\nabla_X Y)^v,\\ &[X^h, Y^h]_Z = -(R(X,Y)Z)^v_Z + [X,Y]^h_Z, \end{aligned} \tag{22}$$

where R denotes the curvature tensor of ∇.

The total space TM has an almost complex structure J determined by ∇. Namely

$$JX^h = X^v, \qquad JX^v = -X^h. \tag{23}$$

From (22) it follows that the almost complex structure is integrable if and only if the connection ∇ is flat.

Assume that ν is a volume form on M such that $\nabla\nu = 0$. In other words, the pair ∇, ν is an equiaffine structure on M. We define a complex volume form Ω on TM by the formula

$$\Omega(X^h_1, ..., X^h_n) = \nu(X_1, ..., X_n). \tag{24}$$

By using (22) one sees that $d\Omega = 0$ if and only if ∇ is flat.

From now on we assume that ∇ is flat and $\nabla\nu = 0$. A manifold with such a structure is usally called an affine manifold with parallel volume. Take the zero-section of TM. The horizontal space at 0_x is equal to T_xM (independently of a connection ∇). Hence the zero-section treated as a mapping $0 : M \to TM$ is an affine Lagrangian embedding. By (24) it is special (also independently of a given connection). We have

Proposition 4.7. *Each affine manifold with parallel volume admits a special affine Lagrangian embedding into a complex space with closed complex volume form.*

From the main theorem of this paper we know that if M is additionally compact, then such embeddings are plentiful.

If ∇ is flat and $\nabla\nu = 0$ then, in fact, the total space of the tangent bundle TM has a structure of a complex equiaffine manifold.

Remark 4.8. Assume now that N is a 4-dimensional almost Kähler manifold with symplectic form κ. Let M be a connected compact orientable 2-dimensional manifold and $f : M \to N$ be a Lagrangian embedding (in the metric sense). We now have the canonical (depending only on the metric) isomorphism, say $\mathfrak{b}$, between vector fields and 1-forms on M. By Theorem 3.4 we have the manifold $\mathcal{M}aL$ modeled on the Fréchet space $\mathcal{C}^\infty(\mathcal{F}^1)$.

Similarly as in the proof of Theorem 4.1 we define the mapping

$$\tilde{P} : \mathcal{C}^\infty(M \leftarrow \mathcal{N}) \ni V \to f_V^*\kappa \in \mathcal{C}^\infty(\mathcal{F}^2). \tag{25}$$

Since $f^*\kappa = 0$ and the normal bundle is star-shaped, the mapping $\tilde{P}$ has values in $\mathcal{C}^\infty(\mathcal{F}^2_{exact})$. By composing the mapping with the isomorphism between the normal bundle and the tangent bundle and the isomorphism $\mathfrak{b}$ we obtain the mapping

$$P : \mathcal{C}^\infty(\mathcal{F}^1) \to \mathcal{C}^\infty(\mathcal{F}^2)$$

whose linearization at 0 is equal to the exterior differential operator d. Now we can argue as in the proof of Theorem 4.1 and we get

Theorem 4.9. *Let N be a 4-dimensional almost Kähler manifold and $f : M \to N$ be a Lagrangian embedding of a connected compact orientable 2-dimensional manifold. Then the set*

$$\mathcal{M}L = \{[f] \in \mathcal{M}aL;\ f \text{ is Lagrangian}\}$$

is an infinite dimensional Fréchet manifold modeled on the Fréchet vector space $\mathcal{C}^\infty(\mathcal{F}^1_{closed})$. It is a submanifold of $\mathcal{M}aL$.

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